On discriminants of the homogeneous polynomial at most four degree (Pursuit of the Essence of Singularity Theory)

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On discriminants of the homogeneous polynomial at most four degree

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1 Introduction

Aim of this report is to discuss the following simple problem:

Problem 1.1. Let

$$F(x,y)=Ax^{4}+Bx^{3}y+Cx^{2}y^{2}+Dxy^{3}+Ey^{4} \tag{1}$$

be a homogeneous polynomial with real value coefficients A, B, C, D, E. Obtain a condition to count the real solutions $[x:y]$ with $f(x,y)=0$ in the real projective space $\mathbb{R}P$ and determine their multiplicities.

We explain the back ground. In [4] we are studying asymptotic directions of spacelike surface in de Sitter space. The asymptotic directions are defined as the kernel directions of the second fundamental form of spacelike surfaces with respect to some special normals, which we call bi-normal directions. The bi-normal directions are given as the solutions of the following trigonometric equation at most two degree:

$$\Delta(\theta)=\sum_{i,j\geq 0,\ i+j\leq 2}c_{i,j}\cos^{i}\theta\sin^{j}\theta=0 \quad \text{where } c_{i,j} \text{ is real coefficient} \tag{2}$$

We obtain the equation (1) to solve the equation (2).

Aim of our study [4] is to classify the second fundamental forms on the spacelike surfaces. The multiplicity of asymptotic directions give important information to classify the surfaces with high co-dimension.

Let $f(x)=F(x,1)$ and denote the degree of $f(x)$ by $\deg f(x)$, then $\deg f(x)\leq 4$ and the multiplicity of the solution $[x:y]=[1:0]$ of $F(x,y)=0$ equals to $4-\deg f(x)$. It is sufficient to discuss the equations $f(x)=0$ for each case $\deg f(x)=0,1,2,3,4$, especially an argument for the case of cubic and quartic equations is interesting.

We remark that the algorithm to obtain the real solutions for $f(x)=0$ of general degree without multiple solutions is known as the Strum's theorem. Even if there are some multiple solutions in $f(x)=0$, we may generalize the above algorithm by eliminating multiple factors of $f(x)$ from the greatest common divisor of $f(x)$ and its derivative $f'(x)$. (see Thomas [3]) On the other hand, the classifications of cubic and quartic equation by their parameters are discussed in [2]

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Some readers may think that it is not significant to discuss the condition for the multiplicity of solutions of the equation $f(x) = 0$. However, the argument on the equations bears some interesting observations from an aspect of singularity theory. For example, the discriminant of cubic equation $x^3+dx+e=0$ is given by $4d^3+27c^2$. and its null set is a *cusp curve*. We also consider the case of quartic equations. Observing Figure.1, a shape of the null set of its discriminant looks like a *swallowtail*, but they are not homeomorphic.

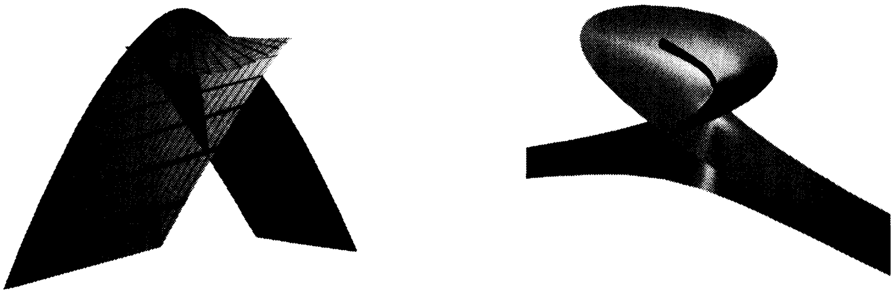

Figure 1: Swallowtail and null set of the discriminant

We argue the discriminants for homogeneous equation $F(x, y) = 0$ to determine the multiplicity of solutions. This argument may gives us an complete condition to know the solutions of the trigonometric equation 2.

We organize this report as follows. In §3 we review the quadratic case as a simple example to make readers to understand the aim of this report. And then we discuss the cubic case in §4 and quartic case in §5. We use the classification of quartic equation in [2], and draw the picture of the discriminants. The classification of the quartic equation is written by three parameters G, H, I. Next, we discuss the discriminant $D^*_{q,1}, I^*, J$ for the homogeneous equation $F(x, y) = 0$. Our attempt is not smart, however we expect that the discriminants helps us to observe the bifurcation of types of equations.

2 Preliminary

We define some notations of the polynomial. Let $f(x)$ be a polynomial with degree n. We say that $f(x)$ or $f(x) = 0$ is $(2m_1+\cdots+2m_k, \ell_1+\cdots+\ell_s)$-type if $2(m_1+\cdots+m_k)+\ell_1+\cdots+\ell_s = n$ and the imaginary solutions of $f(x)$ are $\alpha_i \pm \beta_i\sqrt{-1}$ $(i = 1, \ldots, k)$ with multiplicity m_i and real solutions γ_j $(j = 1, \ldots, s)$ with multiplicity ℓ_j. So $f(x)$ is written by

$$f(x) = ((x-\alpha_1)^2+\beta_1^2)^{m_1}\cdots((x-\alpha_k)^2+\beta_k^2)^{m_k}(x-\gamma_1)^{\ell_1}\cdots(x-\gamma_s)^{\ell_s}$$

If we consider the quadratic case, there are three types: (0,2), (0,1+1) and (2,0).

Let $F(x, y), G(x, y)$ be homogeneous polynomials of real solutions at most four degree, We say that the type of F and G are equivalent if the number of the complex solutions for $F(x,y)=0$, $G(x,y)=0$ and their multiplicities coincide. In this case there are 10 types of homogeneous polynomials (see Table 1)

type	form of $F(x,y)$	type	form of $F(x,y)$
(0,4)	$\pm(a_1x-b_1y)^4$	(0,3+1)	$\pm(a_1x-b_1y)^3(a_2x-b_2y)$
(0,2+2)	$\pm(a_1x-b_1y)^2(a_2x-b_2y)^2$	(0,2+1+1)	$\pm(a_1x-b_1y)^2(a_2x-b_2y)(a_3x-b_3y)$
(0,1+1+1+1)	$\pm(a_1x-b_1y)\cdots(a_4x-b_4y)$	(4,0)	$\pm((a_1x-b_1y)^2+c_1^2y^2)^2$
(2+2,0)	$\pm((a_1x-b_1y)^2+c_1^2y^2)((a_2x-b_2y)^2+c_2^2y^2)$	(2,2)	$\pm((a_1x-b_1y)^2+c_1^2y^2)(a_2x-b_2y)^2$
(2,1+1)	$\pm((a_1x-b_1y)^2+c_1^2y^2)(a_2x-b_2y)(a_3x-b_3y)$	$\sharp S^1$	$F(x,y)\equiv 0$

Table 1: Types of homogeneous polynomial $F(x,y)$. where a_i, b_i, c_i are the real values with $c_i \neq 0$ and $(a_i,b_i)\neq(0,0)$ for all i.

Let m be a positive integer and $F(x,y)$ be a homogeneous polynomial of degree m. We denote $f(x)=F(x,1)$ and $k=\deg f(x)-m$. We say that $f(x)$ is $(2m_1+,\cdots+2m_s, m_1+\cdots+m'_s+\underline{k})$-type if $f(x)$ is $(2m_1+,\cdots+2m_s, m_1+\cdots+m'_s)$-type.

We also review an important tool to consider the equation $f(x)=0$. The discriminant D is given as the resultant of $f(x)$ and its derivative $f'(x)$, which gives us a condition for the existence of the multiple solutions.

3 Quadratic case: Equations of at most two degree

We now review the case $m=2$. Let $f(x)=Cx^2+Dx+E$ a polynomial at most degree. This argument is related to the conjugate class of the symmetric matrices $Sym(2,\mathbb{R})$, and pencils of quadratic forms. (See Bröcker [1])

Suppose that $\deg f=2$ (equivalently $C\neq 0$), then we may classify types of solutions for $f(x)=0$ by using the discriminant $D_{quadratic}=-D^2+4CE$. If $D_{quadratic}$ is positive, negative or equal to zero, then type of $f(x)=0$ is (2,0), (0,1+1), (0,2) respectively. We may also consider the case $\deg f=1$ and $\deg f=0$. (see the following Table3.) If $E=0$ then all real values are solutions of $f(x)=0$. In this case, we say that $f(x)$ is $\sharp S^1$-*type* for convenience.

No.	type	$D_{quadric}$	(C,D,E)	deg$f$
(1)	(2,0)	$+$	$*$	2
(2)	(0,1+1)	$-$	$C\neq 0$	2
(3)	(0,2)	0	$C\neq 0$	2
(2')	(0,1+$\underline{1}$)	$-$	$C=0,\ (C,D,E)\neq \mathbf{0}$	1
(3')	(0,$\underline{2}$)	0	$C=0,\ (C,D,E)\neq \mathbf{0}$	0
(4)	$\sharp S^1$	0	$\mathbf{0}$	0

Table 2: Types of $f(x)=Cx^2+Dx+E=0$ ($*$ means that we do not care the value.)

If $\deg f$ is less than two we should be careful when we classify the types of $f(x)=0$ by using the discriminant $D_{quadratic}$. Fortunately, the discriminant works well when we

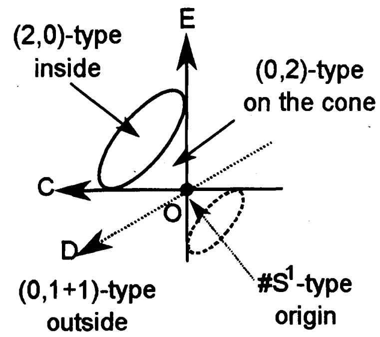

Figure 2: Classification of $F(x,y) = Cx^2 + Dxy + Ey^2 = 0$

consider the types of homogeneous polynomial. Let $F(x,y) = Cx^2 + Dxy + Ey^2$, then we may determine the types of the solutions for $F(x,y) = 0$ by using the same discriminant $D_{quadratic} = -D^2 + 4CE$ except for $\sharp S^1$-case, and this classification does not depend on the degree of the polynomial $f(x) = F(x,1)$. (See Table 3)

No.	type	D	(E,F,G)	position of (E,F,G)
(1)	(2,0)	+	$*$	inside of a cone
(2)	(0,1+1)	$-$	$E \neq 0$	outside of a cone
(3)	(0,2)	0	$E \neq 0$	on a cone
(4)	$\sharp S^1$	0	0	origin

Table 3: Table of types of $F(x,y) = Cx^2 + Dxy + Ey^2 = 0$

The null set of the discriminant consists of a cone $\{(C,D,E) \mid D^2 = CE\}$, which tangents to a C-axis and an E-axis. We may observe that the cone separates the solutions types of the equations.

Finally, we consider the following question:

Question 3.1. Let $F(x,y) = 0$ be a homogeneous polynomial of degree $m \geq 3$. Can we classify the types of $F(x,y) = 0$ by using some discriminant polynomials without the condition for $\deg f$?

The answer of the above question in cubic case is positive, we discuss it in the next section.

4 Cubic case: Equations of at most three degree

In this section we discuss the case of cubic case. If the degree of f is more than two, we may need to introduce the other "discriminant" of the equation $f(x) = 0$. For example, two polynomials $f_1(x) = (x - x_1)^3$ and $f_2(x) = (x - x_1)^2(x - x_2)$ with two distinct real numbers x_1, x_2 are different types, However, both of the discriminants of f_1, f_2 are equal to zero. Therefore, we need another *discriminant* to the cubic case, which we call a *second discriminant.*

4.1 Discriminants for the cubic equation $f(x) = 0$

Let $f(x) = Bx^3 + Cx^2 + Dx + E = 0$. The case when $\deg f < 3$ is already discussed as above. We assume that $B \neq 0$ (equivalently $\deg f = 3$) to apply the classification for the cubic equation. The discriminants of the equation $f(x) = 0$ is

$$D_{cubic,1} = B^2E^2 + \frac{4}{27}(C^3E + BD^3) - \frac{2}{3}BCDE - \frac{1}{27}C^2D^2. \quad (3)$$

$$D_{cubic,2} = 3BD - C^2 \quad (4)$$

By computation, $D_{cubic,1}\,|_{B=0} = -\frac{C^2}{27}(D^2 - 4CE) = \frac{C^2}{27}D_{quadric}$. The second discriminant $D_{cubic,2}$ is used to determine whether there are triple solution for $f(x) = 0$ or not. We remark that the $D_{cubic,2}$ can be replaced to other polynomials. The classification for the cubic equation $f(x) = 0$ is given as follows:

(1) $B \neq 0$ and $D_{cubic,1} < 0$ iff the equation type of $f(x) = 0$ is (0,1+1+1)-type.

(2) $B \neq 0$ and $D_{cubic,1} > 0$ if and only if the equation type is (2,1)-type.

(3) $B \neq 0$, $D_{cubic,1} = 0$ and $D_{cubic,2} \neq 0$ if and only if the equation type is (0,2+1)-type.

(4) $B \neq 0$, $D_{cubic,1} = 0$ and $D_{cubic,2} = 0$ if and only if the equation type is (0,3)-type.

We have the following list of classification for $f(x) = Bx^3 + Cx^2 + Dx + E = 0$, although we may eliminate some cases.

No.	type	B	$D_{cubic,1}$	$C^2D_{quadratic}$	$D_{cubic,2}$	C	D	E	$\deg f$
(1)	(0,1+1+1)	$\neq 0$	$-$	$*$	$*$	$*$	$*$	$*$	3
(2)	(2,1)	$\neq 0$	$+$	$*$	$*$	$*$	$*$	$*$	3
(3)	(0,2+1)	$\neq 0$	0	$*$	$\neq 0$	$*$	$*$	$*$	3
(4)	(0,3)	$\neq 0$	0	$*$	0	$*$	$*$	$*$	3
(5)	(0,1+1+$\underline{1}$)	0	$-$	$= D_{c,1}$	$*$	$\neq 0$	$*$	$*$	2
(6)	(2,$\underline{1}$)	0	$+$	$= D_{c,1}$	$*$	$\neq 0$	$*$	$*$	2
(7)	(0,2+$\underline{1}$)	0	0	$= D_{c,1}$	$BD - C^2 \neq 0$	$\neq 0$	$*$	$*$	2
(8)	(0,$\underline{2}$+1)	0	0	$= D_{c,1}$	$BD - C^2 = 0$	0	$\neq 0$	$*$	1
(9)	(0,$\underline{3}$)	0	0	$= D_{c,1}$	0	0	0	$\neq 0$	0
(10)	$\sharp S^1$	0	0	$= D_{c,1}$	0	0	0	0	0

Table 4: Table of types on $f(x) = Bx^3 + Cx^2 + Dx + E = 0$.

4.2 Second discriminant for the homogeneous case $F(x, y) = 0$

We now consider the types for the homogeneous polynomial $F(x, y) = Bx^3 + Cx^2y + Dxy^2 + Ey^3$. When we try to use the above table for $f(x) = 0$ to the homogeneous case, we may find a problem with (0,2+1) and (0,3) case: the necessity and sufficient condition of the discriminants $D_{cubic,1}, D_{cubic,2}$ for the case (3) and the case (7) listed above agree, but the case (8) does not agree with the condition $D_{cubic,2} \neq 0$. To obtain a new discriminant polynomial instead of $D_{cubic,2}$, we use the following remark.

Remark 4.1. Let $f^I(x) := x^n f(1/x)$ be an inverted polynomial of the polynomial $f(x)$ admitting the case $x = 0$. Then the types of $f(x) = 0$ and $f^I(x) = 0$ coincide.

The above remark implies an idea that *the discriminants should be symmetric under the inversion of variable x*. So that the candidate of the discriminant should be an invariant under the change B,E and C,D. We may easily check that the discriminant D_{cubic1} is symmetric. Inverting $D_{cubic,2} = 3BD - C^2$, we have the inversion $(D_{cubic,2})_{inv} = 3CE - D^2$. Therefore, we obtain a candidate of a *new second discriminant* for the homogeneous polynomial $F(x, y) = 0$:

$$D^{new}_{cubic,2} = (3BD - C^2) + (3CE - D^2). \tag{5}$$

We now show that the discriminants $D_{cubic,1}, D^{new}_{cubic,2}$ work well to classify the types of the homogeneous polynomials $F(x, y) = 0$. If $D_{cubic,1} \neq 0$, we may distinguish (0,1+1+1)-case and (2,1)-case from the other. If $D_{cubic,1} = 0$, we have the following lemma:

Lemma 4.2. Let $F(x, y) = Bx^3 + Cx^2y + Dxy^2 + Ey^3$ and polynomials $D_{cubic,1}$ and $D^{new}_{cubic,2}$ are defined as above. Suppose that $D_{cubic,1} = 0$ then we have:

(1) $F(x, y) = 0$ is (0,2+1)-type if and only if $D^{new}_{cubic,2} \neq 0$.

(2) $F(x, y) = 0$ is (0,3)-type if and only if $D^{new}_{cubic,2} = 0$ and $(B, C, D, E) \neq \mathbf{0}$. ($\sharp S^1$-type corresponds to $(B, C, D, E) = \mathbf{0}$.)

Proof. By assumption, it is sufficient to check the following.

If $f(x) = 0$ is $(0, 2 + 1)$-type, then without loss of generality, we may write $f(x, y) = (\alpha x - \beta y)^2(\gamma x - \delta y)$ with $[\alpha : \beta] \neq [\gamma : \delta] \in \mathbb{R}P^1$. By computation, $D^{new}_{cubic,2} = -(\alpha^2 + \beta^2)(\alpha\delta - \beta\gamma)^2 \neq 0$.

If $f(x) = 0$ is $(0, 3)$-type, then without loss of generality we may write $f(x, y) = (\alpha x - \beta y)^3$, then obtain $D^{new}_{cubic,2} = 0$. □

We finally conclude the list for types of $F(x, y) = 0$, by using two discriminants $D_{c,1} = B^2E^2 + \frac{4}{27}(C^3E + BD^3) - \frac{2}{3}BCDE - \frac{1}{27}C^2D^2$ and $D^{new}_{c,2} = (3BD - C^2) + (3CE - D^2)$.

No.	type	$D_{c,1}$	$D^{new}_{c,2}$	(B,C,D,E)	possibility of deg$f$
(1)	(0,1+1+1)	$-$	*	$\neq 0$	3 or 2
(2)	(2,1)	$+$	*	$\neq 0$	3 or 2
(3)	(0,2+1)	0	$\neq 0$	$\neq 0$	3, 2 or 1
(4)	(0,3)	0	0	$\neq 0$	3 or 0
(5)	$\sharp S^1$	0	0	0	0

Table 5: Types on $F(x,y) = Bx^3 + Cx^2y + Dxy^3 + Ey^4 = 0$

5 Quartic case

5.1 Classification of quartic equation f(x)=0

The classification for the types of solutions in the quartic equation case are also argued some researchers before. In this report, we will use the classification in [2]. Let A, B, C, D and E be real number with $A \neq 0$ and $f(x) = Ax^4 + Bx^3 + Cx^2 + Dx + E$, then all quartic equations are simplified to the following form:

$$f(x) = x^4 + 6Hx^2 + 4Gx + (I - 3H^2) = 0 \tag{6}$$

where $I = \frac{E}{A} - \frac{BD}{4A^2} + \frac{C^2}{12A^2}$, $G = \frac{D}{4A} - \frac{BC}{8A^2} + \frac{B^3}{32A^3}$ and $H = \frac{C}{6A} - \frac{B^2}{16A^2}$. The discriminant of $f(x) = 0$ is written by $D_{q,1} = I^3 - 27(HI - 4H^3 - G^2)^2$.

No.	Type	Condition
(1)	(0,4)	$I = G = H = 0$
(2)	(0,3+1)	$I = G^2 + 4H^3 = 0$ except for $G = H = 0$
(3)	(2,1+1)	$D_{q,1} < 0$
(4)	(0,1+1+1+1)	$I > 0,\ D_{q,1} > 0,\ H < -\frac{\sqrt{I}}{2\sqrt{3}}$
(5)	(2+2,0)	$I > 0,\ D_{q,1} > 0,\ H > -\frac{\sqrt{I}}{2\sqrt{3}}$
(6)	(0,2+1+1)	$I > 0,\ D_{q,1} = 0,\ H < -\frac{\sqrt{I}}{2\sqrt{3}}$
(7)	(0,2+2)	$I > 0,\ D_{q,1} = 0,\ H = -\frac{\sqrt{I}}{2\sqrt{3}}$
(8)	(2,2)	$I > 0,\ D_{q,1} = 0,\ H > -\frac{\sqrt{I}}{2\sqrt{3}}$ except for case (9)
(9)	(4,0)	$I > 0,\ (H,G) = (\frac{\sqrt{I}}{2\sqrt{3}}, 0)$ In this case we have $D_{q,1} = 0$ automatically .

Table 6: Table of types on $f(x) = x^4 + 6Hx^2 + 4Gx + (I - 3H^2) = 0$

We draw a picture of null set of the discriminant $D_{q,1}$ in Figure 1 and Figure 3. We observe that (0,4)-type consists of a curve around (2+2,0)-type domain, we call the curve a (0,4)-*filament*. We review that the (0,4)-type means that there are multiple imaginary solutions of $f(x) = 0$. Since the (0,4)-filament belongs to null set of $D_{q,1}$, we have the following remark.

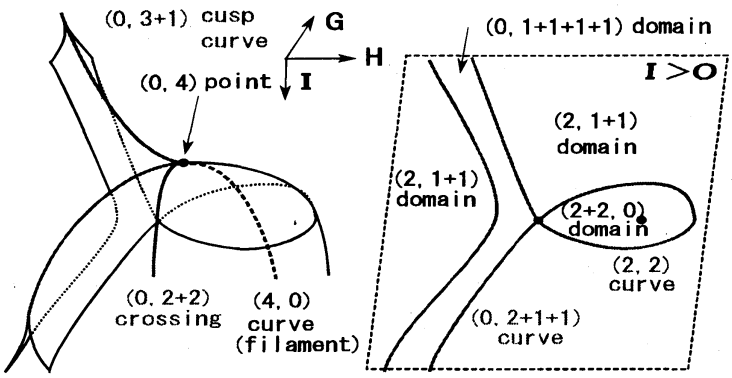

Figure 3: discriminant of the quartic equations

Remark 5.1. Set germ of null set of $D_{q,1}$ at the origin is not homeomorphic to the cuspidal edge. This claim also holds on the parameter space (b, c, d, e) for the quartic equation $f(x) = x^4 + bx^3 + cx^2 + dx + e = 0$.

5.2 Aspects from the invariants

We now switch to argue the homogeneous polynomial and argue the invariants for the quartic equations. Let

$$\Delta(\theta) = \sum_{i,j\geq 0,\ i+j\leq 2} c_{i,j} \cos^i\theta \sin^j\theta = 0$$

be a nontrivial trigonometric equation, by replacing $(\sin\theta, \cos\theta)$ with $(2xy/(x^2+y^2), (y^2-x^2)/(x^2+y^2))$ and multiplying $(x^2+y^2)^2$, we have a homogeneous polynomial of degree at most four $F(x, y) = Ax^4+Bx^3y+Cx^2y^2+Dxy^3+Ey^4$ with real parameters A, B, C, D, E. Suppose that $(A, B, C, D, E) \neq \mathbf{0}$, then we have at most four solutions $[x : y]$ for the equation $F(x, y) = 0$ on the projective complex space $\mathbb{C}P^1$. Let $S = \{Q \in GL(2, \mathbb{R}) \mid \det Q = \pm 1\}$ be a subgroup of general linear group, an action $Q \in S$ to the homogeneous equation $F(x, y) = 0$ is defined by

$$Q.F(x, y) = F(ax + by, cx + dy), \qquad \text{where } Q = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in S.$$

The action $Q \in S$ does not change solution type of the equation $F(x, y) = 0$. We have two invariants I^* and K^* under the action of $Q \in S$:

$$I^* = 12AE - 3BD + C^2, \quad K^* = \frac{1}{2}(72ACE - 27AD^2 - 27B^2E + 9BCD - 2C^3).$$

Let

$$D^*_{q,1} = I^{*\,3} - K^{*\,2}$$

be the discriminant of $F(x, y) = 0$. When $A \neq 0$, the coefficients I, H, G are written by the coefficients A, B, C, D, E of quartic equation $f(x) = F(x, 1) = 0$. We may write $I^* = 12A^2I$, $K^* = 216A^3(HI - 4H^3 - G^2)$ and $D^*_{q,1} = 12^3A^6D_{q,1}$.

Remark 5.2. Let $F(x,y)$ be a homogeneous polynomial and we denote $F_{inv}(x,y) = F(y,x)$ and $F_s(x,y) = F(x,y+sx)$ for real parameter s, then the solution type of $F_{inv}(x,y)=0$ and $F_s(x,y)=0$ are the same as the solution type of $F(x,y)=0$. Moreover, the degree of $F_s(1,x)$, $F_s(x,1)$ is equal to four for almost all s in $\mathbb{R}$.

We denote a candidate of the discriminant:

$$J = \frac{(|A|+|E|)\sqrt{I^*}}{12} + \frac{(AC+CE)}{6} - \frac{B^2+D^2}{16}.$$

It is not a polynomial, perhaps we may have some other discriminants, but we do not argue the existence problem here.

Proposition 5.3. Let $(A,B,C,D,E) \neq 0$ and $F(x,y)=0$ be a homogeneous polynomial of degree four and J is defined as above, then

(1) If the solution type of $F(x,y)=0$ are (0,3+1), (0,1+1+1+1), (0,2+1+1) then $J<0$.

(2) If the solution type of $F(x,y)=0$ are (2+2,0), (2,2), (4,0) then $J>0$.

(3) If the solution type of $F(x,y)=0$ are (0,4), (0,2+2) then $J=0$.

Therefore the sign of J is also an invariant under the actions on S except for (2,1+1) case, and we may discriminate the solution type of $F(x,y)=0$ the as follows:

No.	Type	Condition
(1)	(0,4)	$I^*=0,\ D^*_{q,1}=0,\ J=0,\ (A,B,C,D,E)\neq 0$
(2)	(0,3+1)	$I^*=0,\ D^*_{q,1}=0,\ J<0$
(3)	(2,1+1)	$D^*_{q,1}<0$
(4)	(0,1+1+1+1)	$I^*>0,\ D^*_{q,1}>0,\ J<0$
(5)	(2+2,0)	$I^*>0,\ D^*_{q,1}>0,\ J>0$
(6)	(0,2+1+1)	$I^*>0,\ D^*_{q,1}=0,\ J<0$
(7)	(0,2+2)	$I^*>0,\ D^*_{q,1}=0,\ J=0$
(8)	(2,2)	$I^*>0,\ D^*_{q,1}=0,\ J>0$ except for (9) case
(9)	(4,0)	$I^*>0,\ D^*_{q,1}=0,\ J>0,$ $A\neq 0,\ (H,G)=(\sqrt{I}/(2\sqrt{3}),0)$
(10)	$\sharp S^1$-type ($F(x,y)\equiv 0$)	$(A,B,C,D,E)=0$

Table 7: Table of types on $F(x,y)=Ax^4+Bx^3y+Cx^2y^2+Dxy^3+Ey^4=0$

Therefore, the sufficient and necessity condition that $F(x,y)\geq 0$ (or ≤ 0) for all element $[x:y]$ in $\mathbb{R}P^1$ is $I^*\geq 0$, $D^*_{q,1}\geq 0$ and $J\geq 0$.

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